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A symmetry breaking phenomenon and asymptotic profiles of least energy solutions to a nonlinear Schrödinger equation

東京都立大学理学研究科　渡辺　達也 (Tatsuya Watanabe)
Department of Mathematics,
Tokyo Metropolitan University

1 Introduction

This is a joint work with Kazuhiro Kurata (Tokyo Metropolitan University) and Masataka Shibata (Tokyo Institute of Technology University).

We consider the following nonlinear Schrödinger equation:

$$-u''(x)+(\lambda-\chi_A(x))u(x)=V(x)(1-\chi_A(x))|u|^{p-2}u(x),\ x\in\mathbf{R}, \tag{1}$$

where $\lambda>1$, $p>2$ and χ_A is the characteristic function of a bounded closed interval A. This equation with $p=4$ appears in the study of the propagation of electromagnetic waves through a medium consisting of layers of dielectric materials (see [1], [10]). In this situation, Maxwell's equations for a dielectric medium are the following:

$$\nabla\times\ E=-\frac{1}{c}\frac{\partial B}{\partial t},\ \nabla\times\ H=\frac{1}{c}\frac{\partial D}{\partial t},$$

$$\nabla\cdot D=0,\ \nabla\cdot B=0,$$

where c is the speed of light in a vacuum. The fields E, D, H and B are functions of Cartesian co-ordinates $(x,y,z,t)\in\mathbf{R}^4$.

Assuming that the medium is non-magnetic, i.e. $H\equiv B$, then the remaining constitutive assumption of the medium should determine the displacement field D as a function of the electric field E. We consider the case where the medium is stratified in planes of homogeneous composition perpendicular to the x-axis. In such a medium, we seek solutions of Maxwell's equations with an electric field that are monochromatic of frequency $\omega>0$, propagating along the z-axis and are polarized along the y-axis. A field of this kind is given by

$$E(x,y,z,t)=u(x)e_2\cos(kz-\omega t),$$

where $2\pi/k$ is the wavelength $(k>0)$, $u:\mathbf{R}\to\mathbf{R}$ and e_j $(j=1,2,3)$ are usual basis vectors in $\mathbf{R}^3$.

In circumstances, it is usually assumed that D and E are related by

$$D(x,y,z,t) = (1+4\pi F(x,\frac{1}{2}u(x)^2))E(x,y,z,t).$$

In particular, the most common form of F is

$$F(x,s) = f_1(x) + f_2(x)s \ for \ s \geq 0,$$

where f_1 and f_2 are scalar functions. This form is called the Kerr nonlinearity and is used in various engineering literatures.

Let $n^2(x,s) = 1 + 4\pi F(x,s)$. Then for a magnetic field

$$H = \frac{c}{\omega}(u'(x)\sin(kz-\omega t)e_3 - ku(x)\cos(kz-\omega t)e_1),$$

the problem leads to a second-order nonlinear problem:

$$-u''(x) + k^2 u(x) = \frac{\omega^2}{c^2} n^2(x, \frac{1}{2}u(x)^2)u(x) \ for \ x \in \mathbf{R}.$$

Taking particularly $k^2 = \lambda$ and

$$n^2(x,s) = \begin{cases} \frac{c^2}{\omega^2} & \text{in } A \\ \frac{2c^2}{\omega^2}V(x)s & \text{in } \mathbf{R}\setminus A, \end{cases}$$

we obtain the equation (1) with $p = 4$.

The guidance conditions require that all fields decay to zero as $|x| \to \infty$ and in each plane $y =$ constant, the total electromagnetic energy per unit length in z is finite. This is equivalent to

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x) = 0, \ u(x), u'(x) \in L^2(\mathbf{R}).$$

When $V(x) \equiv 1$ and $A = [-d,d]$ where $2d > 0$ is a thickness of the interval layer, Akhmediev [1] showed that a family of asymmetric solutions bifurcates from the branch of symmetric ones at a certain value $\lambda = \lambda^*$, provided $p = 4$. Ambrosetti, Arcoya and Gámez [2] obtained similar results for any $p > 2$ and small $d > 0$. Arcoya, Cingolani and Gámez [4] showed that a least energy solution is asymmetric for any $d > 0$ by using a variational method. Furthermore, Cingolani and Gámez [5] obtained similar results for the higher dimensinal case.

Our first purpose is to consider a symmetry breaking phenomenon for least energy solutions to (1) with a symmetric multi-layered case. Especially, we consider the case $A = [-l-2,-l] \cup [l, l+2]$ with $l > 0$. In this case, equation is devided into five parts. So we call this the five layered case. Throughout this paper, we assume that

(V0) $V(x) \in C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $V(x) \geq V_0 > 0$ for all $x \in \mathbf{R}$.

Moreover we assume that $V(x)$ is an even function and satisfies the following conditions:

(V1) there exists a limit $V_\infty := \lim_{|x|\to\infty} V(x)$ and $x_0 > 0$ such that $V(x) \geq V_\infty$ for all $|x| \geq x_0$,

(V2) there exists $x_1 \in \mathbf{R}$ such that $V(x_1) := \sup_{x\in\mathbf{R}} V(x)$ and $V_\infty^{-\frac{2}{p}} E_1 < V(x_1)^{-\frac{2}{p}} E_0$ holds for

$$E_0 := \inf_{u\in H^1(\mathbf{R}),\neq 0} \frac{\int_{\mathbf{R}} |u'|^2 + \lambda |u|^2 dx}{(\int_{\mathbf{R}} |u|^p dx)^{\frac{2}{p}}},$$

$$E_1 := \inf_{u\in H^1,\neq 0} \frac{\int_{\mathbf{R}} |u'|^2 + (\lambda - \chi(x))|u|^2 dx}{(\int_{\mathbf{R}} (1-\chi(x))|u|^p dx)^{\frac{2}{p}}},$$

where $\chi(x) = \chi_{[-1,1]}(x)$. This condition (V2) means effect of linear medium is stronger than that of potential $V(x)$.

Our first theorem is the following.

Theorem 1.1. *Let $A = [-l-2, -l] \cup [l, l+2]$. Assume (V0), (V1) and (V2). Then a least energy solution of (1) exists for all $l > 0$ and there exists a sufficiently large constant $l_0 > 0$ such that a least energy solution of (1) is asymmetric for all $l > l_0$.*

Whether the symmetry breaking phenomenon occurs for any $l > 0$ or not is an open problem. It is also a problem that our least energy solution, regarding as a standing wave of time-dependent nonlinear Schrödinger equation, is stable or not. When $V(x) \equiv 1$ and $A = [-d, d]$, stability was studied in [1], [2] and [8]. However in our situation, it is still an open problem.

Our second purpose is to study asymptotic profiles of least energy solutions for the singularly perturbed problem for small $\epsilon > 0$:

$$-\epsilon^2 u''(x) + (\lambda - \chi_A(x))u(x) = V(x)(1-\chi_A(x))|u|^{p-2}u(x), \ x \in \mathbf{R}. \quad (2)$$

We assume that there exists a limit $V_\infty = \lim_{|x|\to\infty} V(x)$ and

(V3) $V(x) \equiv V_\infty$ or there exists $\tilde{x} \in \overline{\mathbf{R} \setminus A}$ such that

$$V(\tilde{x}) = \sup_{x\in\overline{\mathbf{R}\setminus A}} V(x) > V_\infty.$$

When $V(x) \equiv V_\infty$, we take $\tilde{x} = d$. Suppose $V(x)$ is an even function for simplicity. Then we obtain the followings.

Theorem 1.2. *Suppose $A = [-d, d]$ and $V(x)$ is an even function. Assume (V0) and (V3). Let u_ϵ be a least energy solution of (2) and y_ϵ be a maximum point of $u_\epsilon(x)$. Then there exists a subsequence $\epsilon_j \to 0$ such that u_{ϵ_j} has the following asymptotic behavior:*

(i) If $V(d)^{-\frac{2}{p}} E_\alpha < V(\tilde{x})^{-\frac{2}{p}} E_0$, then

$$\sup_{x \in \boldsymbol{R}} |u_{\epsilon_j}(x) - v_1(\frac{x - y_{\epsilon_j}}{\epsilon})| \to 0 \text{ and } \frac{y_{\epsilon_j} - d}{\epsilon_j} \to \alpha.$$

Here $v_1(x)$ is the unique positive solution of

$$-v_1''(x) + (\lambda - \chi_{(-\infty,-\alpha)}(x))v_1(x) = V(d)(1 - \chi_{(-\infty,-\alpha)}(x))v_1^{p-1}(x), \ x \in \boldsymbol{R},$$

$$E_\alpha = \inf_{u \in H^1, \neq 0} \frac{\int_{\boldsymbol{R}} |u'|^2 + (\lambda - \chi_{(-\infty,-\alpha)}(x))|u|^2 dx}{(\int_{\boldsymbol{R}} (1 - \chi_{(-\infty,-\alpha)}(x))|u|^p dx)^{\frac{2}{p}}},$$

and α is the positive constant determined uniquely by p and λ.

(ii) If $V(\tilde{x})^{-\frac{2}{p}} E_0 < V(d)^{-\frac{2}{p}} E_\alpha$, then

$$\sup_{x \in \boldsymbol{R}} |u_{\epsilon_j}(x) - v_2(\frac{x - y_{\epsilon_j}}{\epsilon})| \to 0 \text{ and } y_{\epsilon_j} \to \tilde{x}.$$

Here $v_2(x)$ is the unique positive solution of

$$-v_2''(x) + \lambda v_2(x) = V(\tilde{x})v_2^{p-1}(x), \ x \in \boldsymbol{R}.$$

Remark 1.3. *(i) We can show that $0 < E_\alpha < E_0$. Therefore when $V(x) \equiv V_\infty$, the case (i) occurs.*

(ii) We can obtain similar results even if $A = \cup_{j=1}^N I_j$ with disjoint bounded intervals I_j and $V(x)$ is not an even function.

Although this kind of concentration phenomena was widely studied (see [3], [7], [9], [11]), Theorem 1.2 seems new even for the case $A = [-d, d]$. Especially this problem (2) is closely related to problems with competing potential functions (see [7], [12]):

$$-\epsilon^2 \Delta u(x) + K(x)u(x) = G(x)|u|^{p-2}u(x) \text{ in } \mathbf{R}^n,$$

where K and G are smooth positive functions. In [12], it was proved that a least energy solution concentrates at a global minimum point of:

$$g(x) := \frac{K^{(2p+2n-np)/2(p-2)}(x)}{G^{2/p-2}(x)}.$$

However in our problem, the corresponding function $g(x)$ is not only discontinuous at the boundary of A but also $g(x) = \infty$ in A. Therefore results by [12] can not be applied directly to our case. In our case, Theorem 1.2 says that values of $V(x)$ at the boundary of layers and supremum value of $V(x)$ on $\overline{\mathbf{R} \setminus A}$ determine the location of concentration.

2 Notations

Suppose $A = [-l-2, -l] \cup [l, l+2]$. Corresponding to the equation (1), we define the followings:

$$I_l(u) = \frac{1}{2}\int_{\mathbf{R}} |u'|^2 + (\lambda - \chi_A(x))|u|^2 dx - \frac{1}{p}\int_{\mathbf{R}} V(x)(1-\chi_A(x))|u|^p dx,$$

$$E_l = \inf_{u \in H^1(\mathbf{R}), \neq 0} \frac{\int_{\mathbf{R}} |u'|^2 + (\lambda - \chi_A(x))|u|^2 dx}{(\int_{\mathbf{R}} V(x)(1-\chi_A(x))|u|^p dx)^{\frac{2}{p}}}.$$

$$M_l = \{u \in H^1(\mathbf{R}) \setminus \{0\}; \int_{\mathbf{R}} V(x)(1-\chi_A(x))|u|^p dx = 1\},$$

$$J_l(u) = \int_{\mathbf{R}} |u'|^2 + (\lambda - \chi_A(x))|u|^2 dx \ u \in M_l, \ \tilde{E}_l = \inf_{u \in M_l} J_l(u).$$

E_l is a least energy corresponding to (1). Thus a least energy solution of (1) means a minimizer of E_l. Now we can easily show that for all $u \in H^1(\mathbf{R}) \setminus \{0\}$, there exists a unique $\gamma > 0$ such that $\gamma u \in M_l$. Moreover for such a γ, we have

$$J_l(\gamma u) = \frac{\int_{\mathbf{R}} |u'|^2 + (\lambda - \chi_A(x))|u|^2 dx}{(\int_{\mathbf{R}} V(x)(1-\chi_A(x))|u|^p dx)^{\frac{2}{p}}}, \ u \in H^1(\mathbf{R}) \setminus \{0\}.$$

3 Sketch of the proof of Theorem 1.1

In this section, we give a sketch of the proof of Theorem 1.1. First we show the existence of a least energy solution of (1). To this aim, we need the following lemmas.

Lemma 3.1. *Suppose (V0) and that there exists a limit* $V_\infty = \lim_{|x| \to \infty} V(x)$. *Then for all* $l > 0$, $I_l(u)$ *satisfies Palais-Smale condition on a sublevel*

$$\Sigma := \{u \in H^1(\boldsymbol{R}); I_l(u) < \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_0)^{\frac{p}{p-2}}\}.$$

We omit the proof of Lemma 3.1, since it is rather standard (cf [4]).

Lemma 3.2. *Assume (V1) holds. Then for all* $l > 0$,

$$\tilde{E}_l = \inf_{u \in M_l} J_l(u) < V_\infty^{-\frac{2}{p}} E_0.$$

Proof. We follw arguements in [4]. Fix $l > 0$. Suppose $z(x)$ is a minimizer of

$$\inf_{u \in H^1(\mathbf{R}), \neq 0} \frac{\int_{\mathbf{R}} |u'|^2 + \lambda |u|^2 dx}{(\int_{\mathbf{R}} V_\infty |u|^p dx)^{\frac{2}{p}}},$$

and put $z_\theta(x) = z(x+\theta)$. Then there exists a unique $\gamma_\theta > 0$ such that $\gamma_\theta z_\theta \in M_l$. We show that $J_l(\gamma_\theta z_\theta) < V_\infty^{-\frac{2}{p}} E_0$ for large θ. Now we have

$$J_l(\gamma_\theta z_\theta) = \frac{\int_{\mathbf{R}} |z_\theta'|^2 + \lambda z_\theta^2 dx - \int_{\mathbf{R}} \chi_A(x) z_\theta^2 dx}{(\int_{\mathbf{R}} V(x) z_\theta^p dx - \int_{\mathbf{R}} V(x)\chi_A(x) z_\theta^p dx)^{\frac{2}{p}}}$$

$$= \frac{\int_{\mathbf{R}} |z'(t)|^2 + \lambda z(t)^2 dt - \int_{\theta+l}^{\theta+l+2} z(t)^2 dt - \int_{\theta-l-2}^{\theta-l} z(t)^2 dt}{(\int_{\mathbf{R}} V(t-\theta) z(t)^p dt - \int_{\theta+l}^{\theta+l+2} V(t-\theta) z(t)^p dt - \int_{\theta-l-2}^{\theta-l} V(t-\theta) z(t)^p dt)^{\frac{2}{p}}}, \tag{3}$$

where we put $x + \theta = t$.

Since $z(t)$ is a solution of

$$-z''(t) + \lambda z(t) = V_\infty z(t)^{p-1}, \ t \in \mathbf{R},$$

we have

$$z(t) e^{\sqrt{\frac{\lambda}{2}}|t|} \to c > 0 \ (|t| \to \infty).$$

Thus for all $\delta > 0$, there exists $r > 0$ such that for all $|t| \geq r$,

$$(c-\delta) e^{-\sqrt{\frac{\lambda}{2}}|t|} \leq z(t) \leq (c+\delta) e^{-\sqrt{\frac{\lambda}{2}}|t|}.$$

Therefore for large θ, we have

$$\int_{\theta+l}^{\theta+l+2} V(t-\theta) z(t)^p dt \leq c_1 \|V\|_{L^\infty} e^{-p\sqrt{\frac{\lambda}{2}}|\theta+l|},$$

$$\int_{\theta+l}^{\theta+l+2} z(t)^2 dt \geq c_2 e^{-2\sqrt{\frac{\lambda}{2}}|\theta+l|},$$

where c_1, c_2 are constants independent on θ. From (3), we have

$$J_l(\gamma_\theta z_\theta) \leq \frac{\int_{\mathbf{R}} |z'|^2 + \lambda |z|^2 dt - c_3 e^{-2\sqrt{\frac{\lambda}{2}}\theta}}{(\int_{\mathbf{R}} V(t-\theta) z^p dt - c_4 e^{-p\sqrt{\frac{\lambda}{2}}\theta})^{\frac{2}{p}}}$$

$$= \frac{\int_{\mathbf{R}} |z'|^2 + \lambda |z|^2 dt}{(\int_{\mathbf{R}} V(t-\theta) z^p dt)^{\frac{2}{p}}} \times \frac{1 - c_5 e^{-2\sqrt{\frac{\lambda}{2}}\theta}}{(1 - c_6 e^{-p\sqrt{\frac{\lambda}{2}}\theta})^{\frac{2}{p}}}$$

$$= \frac{\int_{\mathbf{R}} |z'|^2 + \lambda |z|^2 dt}{(\int_{\mathbf{R}} V(t-\theta) z^p dt)^{\frac{2}{p}}} \times \frac{1 - c_5 e^{-2\sqrt{\frac{\lambda}{2}}\theta}}{1 - \frac{2}{p} c_6 e^{-p\sqrt{\frac{\lambda}{2}}\theta} + o(e^{-p\sqrt{\frac{\lambda}{2}}\theta})}.$$

Since $p > 2$, for sufficiently large θ, it follows that

$$\frac{1 - c_5 e^{-2\sqrt{\frac{\lambda}{2}}\theta}}{1 - \frac{2}{p} c_6 e^{-p\sqrt{\frac{\lambda}{2}}\theta} + o(e^{-p\sqrt{\frac{\lambda}{2}}\theta})} < 1.$$

Moreover by (V1), for large θ, $V(t-\theta) \geq V_\infty$. Therefore for sufficiently large θ, we have

$$J_l(\gamma_\theta z_\theta) < \frac{\int_{\mathbf{R}} |z'|^2 + \lambda |z|^2 dt}{(\int_{\mathbf{R}} V_\infty z^p dt)^{\frac{2}{p}}} = V_\infty^{-\frac{2}{p}} E_0.$$

□

Once we obtain Lemma 3.1 and 3.2, we can show the existence of a positive least energy solution $u_l(x)$ to (1) by the standard variational method. The key of the proof of Theorem 1.1 is the following lemma.

Lemma 3.3. *Assume (V1) holds. Then for sufficiently large l,*

$$\tilde{E}_l < V_\infty^{-\frac{2}{p}} E_1.$$

The proof can be done as in Lemma 3.2. This lemma means that if each linear mediums are very far, the least energy of five layered case is less than that of three layered case.

Sketch of the proof of Theorem 1.1: Now we show a contradiction if we assume $u_l(x)$ is symmetric. Let x_l be a maximum point of $u_l(x)$. Then we can easily show that $x_l \in \mathbf{R} \setminus A$. Without loss of generality, we may assume that $x_l \geq 0$. We distinguish into three cases: (i) $x_l > l + 2$, (ii) $0 \leq x_l < l$ and $l - x_l \to \infty$ $(l \to \infty)$, (iii) $0 \leq x_l < l$ and $l - x_l \to \alpha > 0$ $(l \to \infty)$ for some α.

Although we omit the details, we have the following energy estimates.

Estimate for case (i):

$$\frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_0)^{\frac{p}{p-2}} \leq \liminf_{l\to\infty} I_l(u_l) \leq \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_1)^{\frac{p}{p-2}} < \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_0)^{\frac{p}{p-2}}.$$

Estimate for case (ii):

$$\frac{p-2}{2p}(V(x_1)^{-\frac{2}{p}} E_0)^{\frac{p}{p-2}} \leq \liminf_{l\to\infty} I_l(u_l)$$

$$\leq \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_1)^{\frac{p}{p-2}} < \frac{p-2}{2p}(V(x_1)^{-\frac{2}{p}} E_0)^{\frac{p}{p-2}}.$$

Estimate for case (iii):

$$2 \times \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_1)^{\frac{p}{p-2}} \le \liminf_{l\to\infty} I_l(u_l) \le \frac{p-2}{2p}(V_\infty^{-\frac{2}{p}} E_1)^{\frac{p}{p-2}}.$$

We emphasize that we use the assumption (V2) to obtain the estimate of the case (ii). In any case, we obtain a contradiction. This completes the sketch of the proof of Theorem 1.1.

4 Sketch of the proof of Theorem 1.2

Hereafter we consider the singularly perturbed problem for small $\epsilon > 0$:

$$-\epsilon^2 u(x) + (\lambda - \chi_A(x))u(x) = V(x)(1-\chi_A(x))|u|^p u(x), \; x \in \mathbf{R}. \quad (4)$$

Suppose $A = [-d, d]$ and $V(x)$ is an even function for simplicity.

In a similar way as in Theorem 1.1, we can obtain a positive least energy solution $u_\epsilon(x)$ of (4) under the assumptions (V0) and (V3). We denote by y_ϵ a maximum point of $u_\epsilon(x)$. Without loss of generality, we may assume that $y_\epsilon \ge 0$. Putting $u_\epsilon(x) = v_\epsilon(\frac{x-y_\epsilon}{\epsilon})$, we get the following equation:

$$-v_\epsilon''(x) + (\lambda - \chi_A(\epsilon x + y_\epsilon))v_\epsilon(x) = V(\epsilon x + y_\epsilon)(1-\chi_A(\epsilon x + y_\epsilon))v_\epsilon^{p-1}(x), \; x \in \mathbf{R}.$$

Then $v_\epsilon(x)$ is uniformly bounded in $H^1(\mathbf{R})$. Thus we may assume that $v_\epsilon(x) \rightharpoonup v(x)$ in $H^1(\mathbf{R})$ for some $v(x) \in H^1(\mathbf{R})$. Then $v(x) \not\equiv 0$ because $v_\epsilon(0) \ge (\frac{\lambda}{\|V\|_{L^\infty}})^{\frac{1}{p-2}}$. The following proposition is most important.

Proposition 4.1. *(i) Assume (V3) and $V(d)^{-\frac{2}{p}} E_\alpha < V(\tilde{x})^{-\frac{2}{p}} E_0$. Then $\frac{y_\epsilon - d}{\epsilon} \to \alpha > 0$ for some α and $v(x)$ satisfies*

$$-v''(x) + (\lambda - \chi_{(-\infty,-\alpha)}(x))v(x) = V(d)(1-\chi_{(-\infty,-\alpha)})v^{p-1}(x), \; x \in \boldsymbol{R}. \quad (5)$$

(ii) Assume (V3) and $V(\tilde{x})^{-\frac{2}{p}} E_0 < V(d)^{-\frac{2}{p}} E_\alpha$. Then $y_\epsilon \to \tilde{x}$ and $v(x)$ satisfies

$$-v''(x) + \lambda v(x) = V(\tilde{x}) v^{p-1}(x), \; x \in \boldsymbol{R}. \quad (6)$$

This proposition says when $V(d)^{-\frac{2}{p}} E_\alpha < V(\tilde{x})^{-\frac{2}{p}} E_0$, the maximum point y_ϵ goes to the boundary of layer d.

Proof. We consider the case (i). First we show that $\frac{y_\epsilon - d}{\epsilon} \to \alpha > 0$. If $\frac{y_\epsilon - d}{\epsilon} \to \infty$, then $v(x)$ satisfies (6). In fact, for all $\varphi \in C_0^\infty(\mathbf{R})$, we have

$$\int_{\mathbf{R}} v_\epsilon' \varphi' + (\lambda - \chi_A(\epsilon x + y_\epsilon))v_\epsilon \varphi dx = \int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1-\chi_A(\epsilon x + y_\epsilon))v_\epsilon^{p-1} \varphi dx. \quad (7)$$

Then

$$\int_{\mathbf{R}} \chi_A(\epsilon x + y_\epsilon) v_\epsilon \varphi dx = \int_{\frac{-y_\epsilon - d}{\epsilon}}^{\frac{d-y_\epsilon}{\epsilon}} v_\epsilon \varphi dx \to 0 \ (\epsilon \to 0).$$

Moreover since $v_\epsilon(x)$ is a least energy solution, we have $y_\epsilon \to \tilde{x}$. Then we have

$$\int_{\mathbf{R}} v'\varphi' + \lambda v\varphi dx = \int_{\mathbf{R}} V(\tilde{x}) v^{p-1} \varphi dx,$$

that is, $v(x)$ satisfies (6). Thus we have

$$\liminf_{\epsilon \to 0} \Big(\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1 - \chi_A(\epsilon x + y_\epsilon)) v_\epsilon^p dx\Big)^{\frac{p-2}{p}} \geq V(\tilde{x})^{-\frac{2}{p}} E_0.$$

On the other hand, since $v_\epsilon(x)$ is the least energy solution, we have

$$\Big(\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1 - \chi_A(\epsilon x + y_\epsilon)) v_\epsilon^p dx\Big)^{\frac{p-2}{p}}$$

$$\leq \frac{\int_{\mathbf{R}} |u'|^2 + (\lambda - \chi_A(\epsilon x + y_\epsilon))|u|^2 dx}{\left(\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1 - \chi_A(\epsilon x + y_\epsilon))|u|^p dx\right)^{\frac{2}{p}}} \ \textit{for all } u \in H^1(\mathbf{R}) \setminus \{0\}.$$

Choosing a suitable test function, we obtain

$$\liminf_{\epsilon \to 0} \Big(\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1 - \chi_A(\epsilon x + y_\epsilon)) v_\epsilon^p dx\Big)^{\frac{p-2}{p}} \leq V(d)^{-\frac{2}{p}} E_\alpha.$$

This is a contradiction to the assumption. Therefore $\frac{y_\epsilon - d}{\epsilon} \to \alpha$. Tending $\epsilon \to 0$ in (7), we have

$$\int_{\mathbf{R}} v'\varphi' + (\lambda - \chi_{(-\infty,-\alpha)}(x)) v\varphi dx = \int_{\mathbf{R}} V(d)(1 - \chi_{(-\infty,-\alpha)}(x)) v^{p-1} \varphi dx,$$

that is, $v(x)$ satisfies (5).

In the similar arguement as (i), claims of (ii) and (iii) follow. □

Next we consider the equation:

$$-u''(x) + (\lambda - \chi_{(-\infty,-\alpha)}(x)) u(x) = (1 - \chi_{(-\infty,-\alpha)}(x))|u|^{p-2} u(x), \ x \in \mathbf{R}, \quad (8)$$

We can obtain a positive solution of (8) by the standard variational method. Moreover by the phase plane analysis, we can determine α uniquely and obtain the uniqueness of positive solution $v_1(x)$ to (8).

Finally, we show that $v_\epsilon(x)$ has only one peak for small $\epsilon > 0$.

Proposition 4.2: *Assume (V3). Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $v_\epsilon(x)$ has only one peak at the origin.*

Sketch of the proof of Proposition 4.2. We consider the case $V(d)^{-\frac{2}{p}}E_\alpha < V(\tilde{x})^{-\frac{2}{p}}E_0$.

Notice that $v_\epsilon(x)$ has a local maximum point at the origin. Assume there exists $z_\epsilon \neq 0$ such that z_ϵ is a local maximum point of $v_\epsilon(x)$. Since $v_\epsilon(x) \to v_1(x)$ in $C^1_{loc}(\mathbf{R})$ and $v_1(x)$ has a local maximum point only at the origin, it follows $|z_\epsilon| \to \infty$ $(\epsilon \to 0)$. Moreover, we can show that $\epsilon z_\epsilon \to 0$. Then we obtain the following energy estimate:

$$\liminf_{\epsilon\to 0} \frac{p-2}{2p}\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1-\chi_A(\epsilon x + y_\epsilon))v_\epsilon^p dx \geq \frac{3}{2}I_\alpha(v_1) > 0,$$

where $I_\alpha(v_1)$ is the energy corresponding to (8).

On the other hand, since $v_\epsilon(x)$ is the least energy solution, we have

$$\liminf_{\epsilon\to 0} \frac{p-2}{2p}\int_{\mathbf{R}} V(\epsilon x + y_\epsilon)(1-\chi_A(\epsilon x + y_\epsilon))v_\epsilon^p dx \leq I_\alpha(v_1).$$

This is a contradiction. Therefore $v_\epsilon(x)$ has a unique local maximum point at the origin. □

By Proposition 4.2, we can show that $v_\epsilon(x)$ converges to $v(x)$ strongly in $H^1(\mathbf{R})$ and uniformly in $C^0(\mathbf{R})$ by Sobolev's Imbedding Theorem. Therefore the claims of Theorem 1.2 follow. This completes the sketch of the proof of Theorem 1.2.

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